Risp 7: Teacher Notes

Suggested Use: to introduce the idea of implicit differentiation

The volume, surface area and total edge-length of a cube and the ways in which they can be connected are something of a leitmotif in my risps. They seem a natural thing to calculate when faced with a solid, and the way you come across something linear, quadratic and then cubic in turn is often helpful. Even the following beautifully simple question is a rich one:

There are six ways to write V, S and E in order of size.
Can you find a cube for each order?
What about a cuboid?

(with thanks to Rachel Bolton)

So how did I start on this risp? I began by asking students to think a bit about the problem. I watched some useful work: some students used the hint to arrive at the fact that:

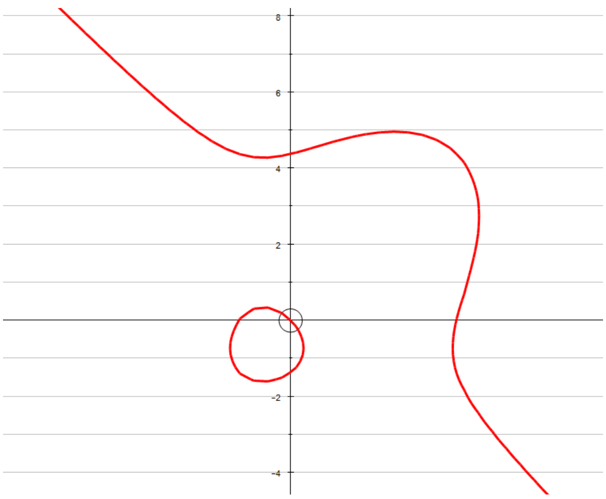
$$E = 12x + 12y,\ S = 6x^2 + 6y^2, \text{ and } V = x^3 + y^3.$$

So one of:

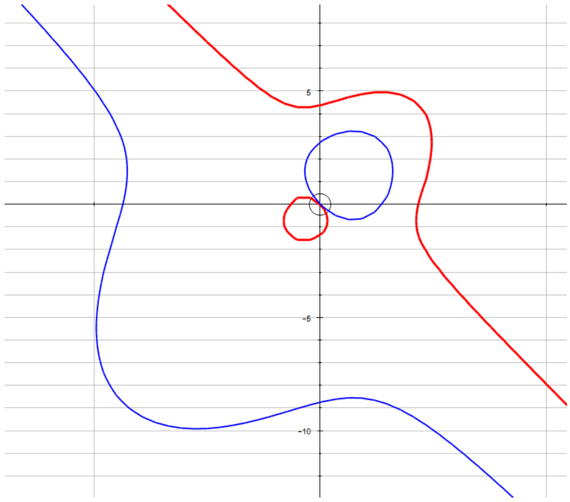
$$12x + 12y + 6x^2 + 6y^2 = 2x^3 + 2y^3$$
$$6x^2 + 6y^2 + x^3 + y^3 = 24x + 24y$$
$$12x + 12y + x^3 + y^3 = 12x^2 + 12y^2$$

must be true. Beyond this, what could they do? Time to introduce a graphing package, one with a facility for drawing implicitly-defined functions.

What would $12x + 12y + 6x^2 + 6y^2 = 2x^3 + 2y^3$ look like? I would defy anyone to sketch this without computer help, although it would be worth noting that we do expect a graph that is symmetrical in y = x.

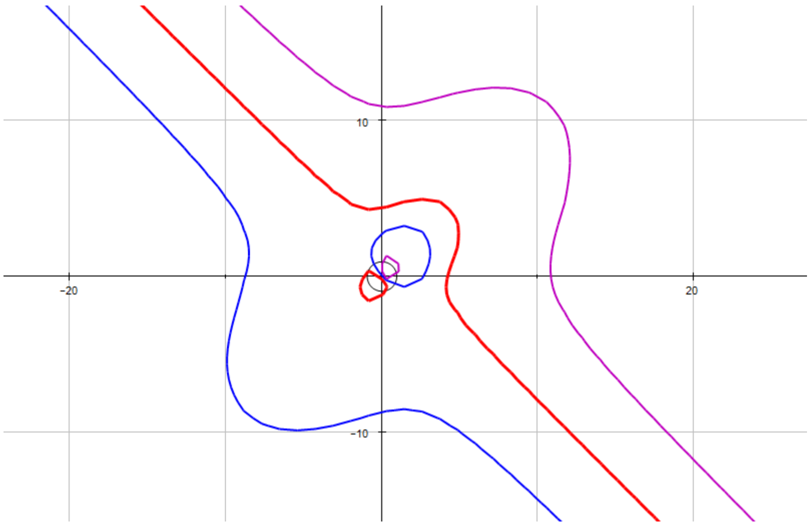


My students reacted with delight to this, as something unexpected and offbeat. So which area of the diagram are we interested in? My students quickly saw that the first quadrant is the answer to this - x and y must be positive. So we do have a maximum to find here. But might the other equations yield something larger? Adding the graph of $6x^2 + 6y^2 + x^3 + y^3 = 24x + 24y$ gives the following:



So another maximum in the first quadrant here, but not a larger one. We can now add the third curve, $12x + 12y + x^3 + y^3 = 12x^2 + 12y^2$.

So this shows us the value of x we seek. We can home in on the stationary point, but the rather grainy drawing of this implicit graph limits the accuracy we can obtain (fortunately, given that we are heading towards finding a really accurate solution by differentiating.) By inspection we arrive at 12.1... for y and 7.46... for x.



So, how would we differentiate to find the maximum point in this case? We can see the point we want to find, but we have never tried to find y' for something like this, where we cannot make y the subject. A perfect opportunity has arisen to introduce implicit differentiation. If we carry out the procedure here we arrive at:

$$y' = (8x - x^2 - 4)/(4 + y^2 - 8y)$$

$$\text{So } y' = 0 \text{ if } x^2 - 8x + 4 = 0, \text{ or } x = 7.464..., \text{ or } 0.5358...$$

We can see the two maxima for these x-values on the graph.

So the x we seek is $4 + \sqrt{12} = 7.464...$, and to find the corresponding y-value, we substitute back into * getting:

$$y^3 = 12y^2 - 12y + 163.138...$$

Happily, the obvious rearrangement yields us the root after iteration:

$$y = \sqrt[3]{(12y^2 - 12y + 163.138...)} \text{ tends to } y = 12.12043946....$$

So we end up with a value for x of 7.46410 (6sf) and for y of 12.1204 (6sf).

Let me mention the word 'scaffolding', a term introduced into educational literature by Wood, Bruner and Ross in 1976. The connotations are of support while building is in progress, but with the obvious rider that the support is intended to be temporary. Here I found that a lot of scaffolding was necessary at the start, but that it could be swiftly dismantled.

This risp uses the fact that motivated theory is remembered theory. The cube problem carried my group along to the point where the idea that we would need the maximum of an implicitly-defined function seemed quite natural. They were then able to approach more prosaic questions with equanimity.

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