Risp 40: Teacher Notes

Suggested use: to consolidate/revise co-ordinate geometry, curve-sketching

This last risp is perhaps the least syllabus-based of them all, a final bit of fun. It provides an entry point for a discussion about the equations of conics. To demonstrate that any bit of mathematics can be enriched (enrisped?) when viewed in the right way, I set myself the challenge of turning my Risp logo into a risp itself.

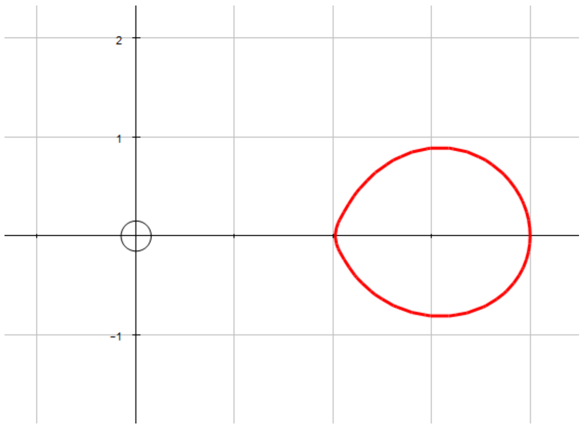
The best way into this problem is to set up Cartesian axes with the origin at R, with S at (2, 0) and with P at (3, 0). Then it is not too far to the following equation:

$$\sqrt{(x^2 + y^2)} + 2 + \sqrt{((x - 2)^2 + y^2)} = 2(\sqrt{((x - 2)^2 + y^2)} + 1 + \sqrt{((x - 3)^2 + y^2)}.$$

This simplifies to:

$$\sqrt{(x^2 + y^2)} = \sqrt{((x - 2)^2 + y^2)} + 2\sqrt{((x - 3)^2 + y^2)}.$$

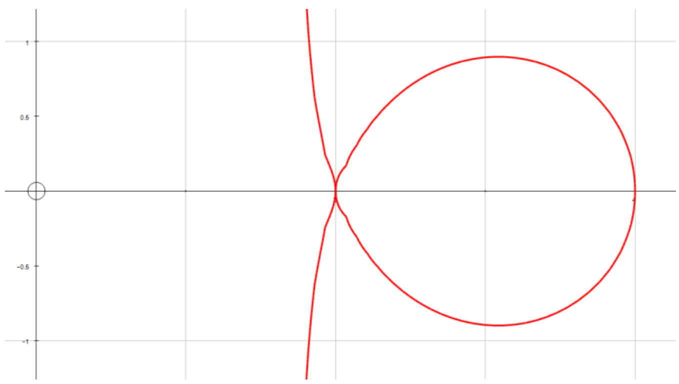
This can happily be graphed implicitly:.



If we try to eliminate the square roots in this equation, we eventually arrive at:

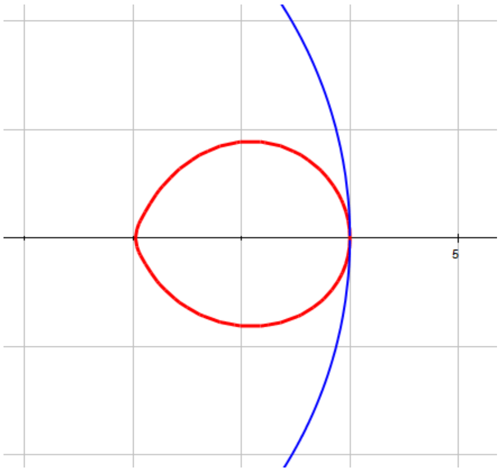
$$4x^3 + 4xy^2 - 32x^2 - 7y^2 + 80x - 64 = 0.$$

Graphing this gives the following:



The graphing gets approximate near (2, 0)! The extra bits of curve arise through our squaring.

When is QR a maximum? Drawing $x^2 + y^2 = 16$ gives the answer: QR = 4 when y = 0.



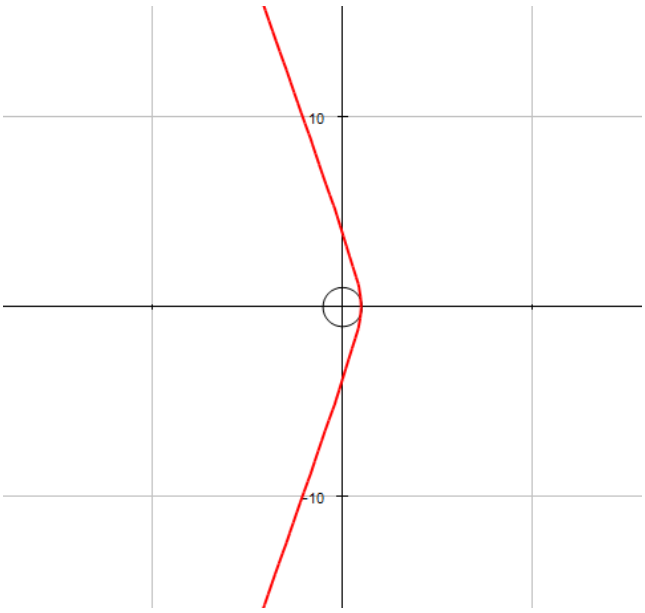
When we replace the number 2 with k in the problem, we get the following equation:

$$\sqrt{x^2 + y^2} + 2 - k + (1 - k)\sqrt{(x - 2)^2 + y^2} - k\sqrt{(x - 3)^2 + y^2} = 0.$$

Trying to get rid of the square roots this time is not recommended! So what happens here as we vary k? Increasing k from 2 gives a curve of smaller and smaller area based around (3, 0). When k = 3, this curve disappears. If we decrease k from 2 towards 1, the area of the curve gets larger and larger, looking almost circular for a while before becoming reminiscent of a cardioid. Always, however, the largest value of QR happens where y = 0 on the curve.

Putting y = 0 into $\sqrt{x^2 + y^2} + 2 - k + (1 - k)\sqrt{(x - 2)^2 + y^2} - k\sqrt{(x - 3)^2 + y^2} = 0$ gives (eventually) $x = \dfrac{2k}{k-1}$ as the maximum value for QR.

Clearly something will happen here when k = 1. Suddenly the curve appears to become unbounded.



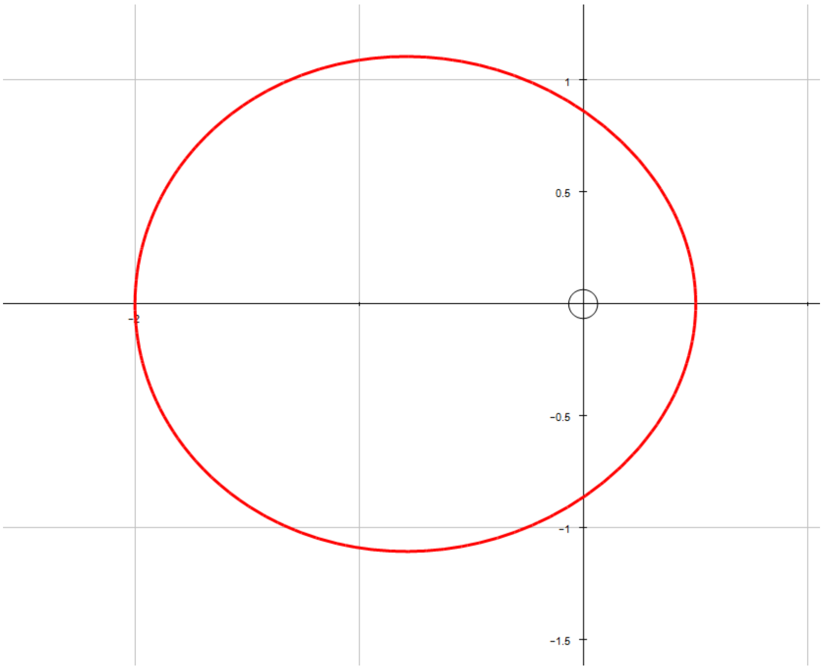
If we try to find the curve's equation, we get:

$$\sqrt{(x^2 + y^2)} + 1 + -\sqrt{((x-3)^2 + y^2)} = 0.$$

Removing the square roots (everything cancels obligingly) gives:

$$\left(x - \frac{13}{4}\right)^2 - \frac{y^2}{4} = \frac{251}{48}.$$

So we have a hyperbola, and so QR can be as large as we wish if k = 1. (We only want one half of the hyperbola: the other half has again been introduced by squaring.) Reducing k further, we arrive at the egg-shaped curve below, based around the origin.



How far can we reduce k? When k = 2/3, the curve disappears once more.

One last point; it is worth noting that throughout this risp, the area of RSQ is twice the area of SPQ.

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