Risp 33: Teacher Notes

Suggested use: to consolidate/revise hyperbolics, exponentials, percentage error, curve sketching

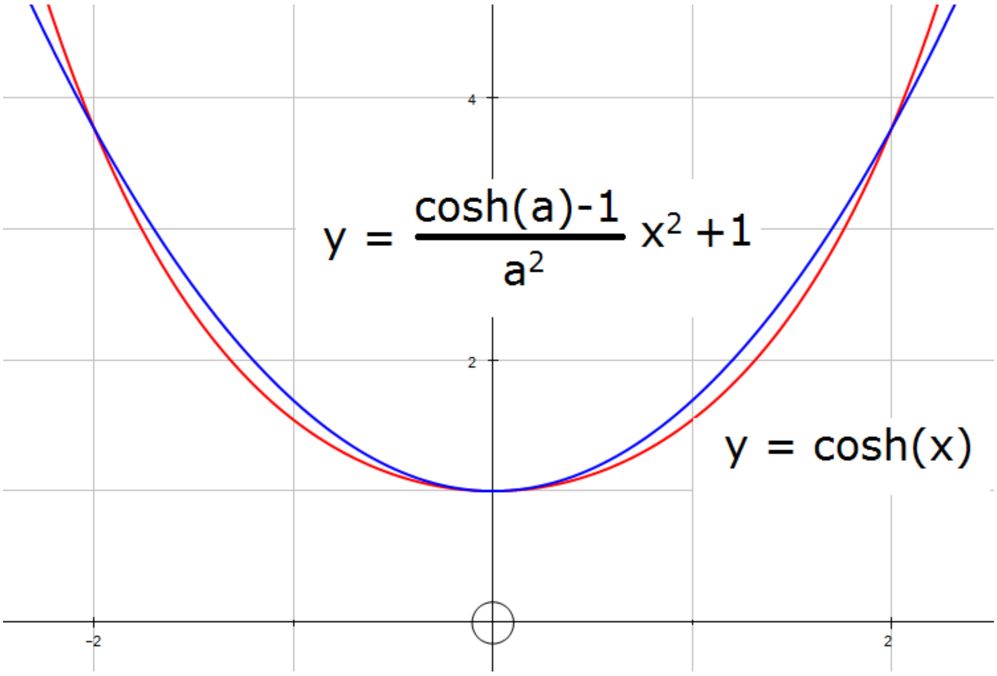
When asked what curve a chain takes up when left to hang naturally, most students will guess at a parabola. This is not far from the truth, but how far exactly? This risp invites students to try to put a number to it. Historically speaking, Galileo claimed in around 1640 that a flexible chain would fall into a parabola – it was Huygens in 1646 (at the age of just 17) who showed this could not be true. The famous mathematical intermediary of the age Mersenne was most impressed. In 1691 Huygens returned to the problem to calculate the true shape of the curve, a catenary given by the hyperbolic cosh function. This is on the Further Maths syllabus, but students not taking Further should be able to handle cosh(x) = $\frac{e^x + e^{-x}}{2}$.

Let's take the general case first: so the catenary has equation y = cosh x.

If the parabola has equation $y = px^2 + 1$, and (a, cosh a) is on the curve,

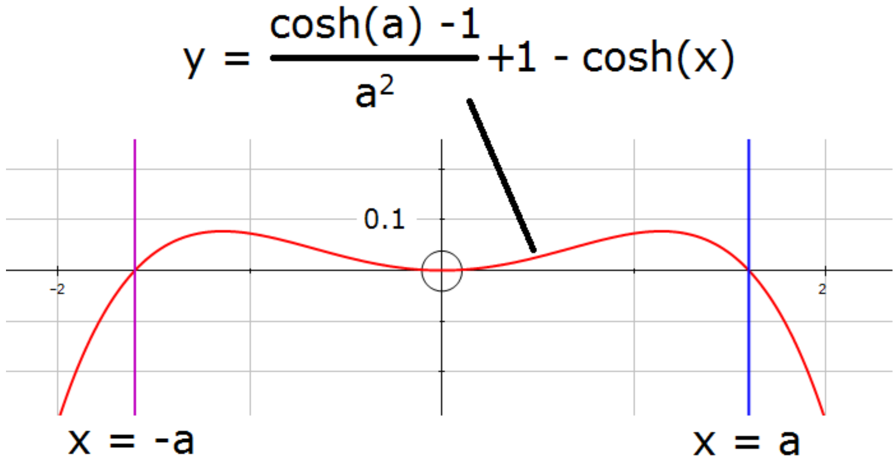
then $\cosh a = pa^2 + 1$, so $p = \frac{\cosh a - 1}{a^2}$.

So the equation of the parabola is $y = \frac{\cosh a - 1}{a^2}x^2 + 1$.



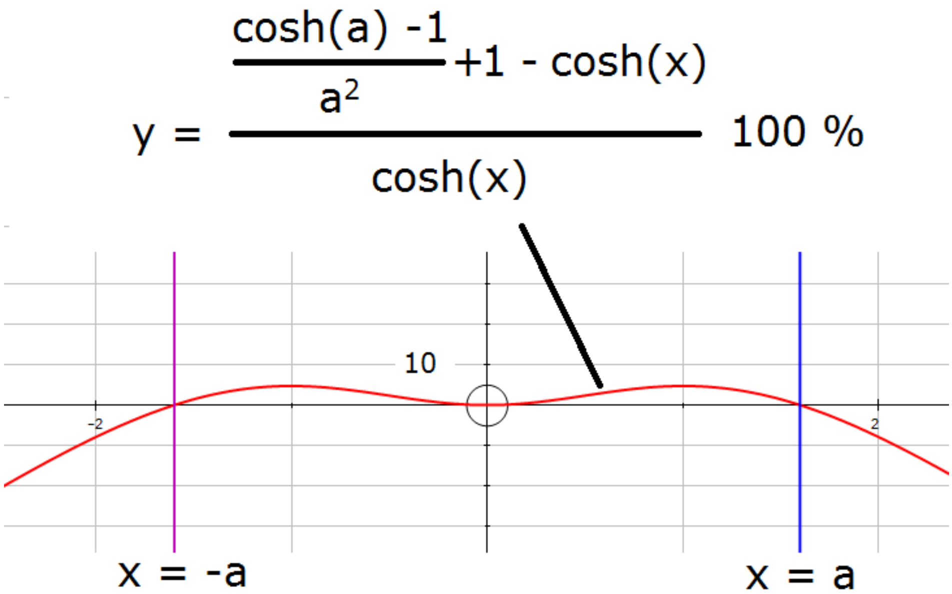
This means the difference between the curves at x is $\frac{\cosh a - 1}{a^2}x^2 + 1$ - cosh x.

For x between a and –a, we can see that the blue line (the parabola) is above the red line (the catenary) so this will be positive. The dominant term here is e^x as x gets bigger.



The percentage error will be $\dfrac{\dfrac{\cosh a - 1}{a^2}x^2 + 1 - \cosh x}{\cosh x} \times 100\%$.

Here again cosh x dominates.



It is tempting to try to differentiate here, but it is quite a lot of hassle for not much reward. Sketching these curves in Autograph and seeing what happens as a varies is instructive and easier.

To answer the question set in the risp, when a = 1, the maximum difference is around 0.010949 when x = 0.71..., while the largest % error is around 0.87723% when x = 0.67... As a increases, these values increase rapidly.

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