Risp 30: Teacher Notes

Suggested Use: to consolidate/revise differential equations
(variables separable only)

Writing risps can be a great pleasure. One has a good idea to start with, where the main body of the problem is clear, then the challenge is to come up with a final line that ties the ends together neatly. Here is an example where after weeks of trying possibilities, the punch-line eventually suggested itself. I wanted (0, 0) and (0, 1) to lie on all solutions, and this implied that the quadratic would need to be of the form $ax^2 - ax + b$.

The Mix-Them-All-Up technique produces just three distinct differential equations here, but each requires a different method of solution. There is $\int f(y)dy = \int g(x)h(x)\,dx$ with f, g and h being polynomials, where it is necessary to multiply out g(x)h(x). Secondly, there is the need to recognise $\int f'(x)/f(x)\,dx$, which gives a natural logarithm. And thirdly, there is the need to integrate g(x)/h(x) where this is improper. Algebraic long division here leads to a polynomial plus a natural logarithm. The idea with students is to address particular cases, but below I tackle the case in general.

Differential Equation 1

$$(ax^2 - ax + b)\frac{dy}{dx} = (cy + d)(2ax - a)$$

$$\Rightarrow \int \frac{1}{cy+d}\,dy = \int \frac{2ax-a}{ax^2-ax+b}\,dx.$$

$$\Rightarrow \frac{1}{c}\ln|cy + d| = \ln|ax^2 - ax + b| + k.$$

$$(0, 0) \text{ on the curve} \Rightarrow \frac{1}{c}\ln|d| = \ln|b| + k \Rightarrow k = \ln\frac{|d|^{1/c}}{|b|}$$

$$\Rightarrow \text{particular solution is } \frac{1}{c}|cy + d| = \frac{|d|^{1/c}}{|b|}|ax^2 - ax + b|.$$

It is easy to check that (1, 0) is on the curve.

Differential Equation 2

$$(cy + d)\ \frac{dy}{dx} = (ax^2 – ax + b)(2ax – a).$$

$$\Rightarrow cy^2 + dy = \int 2a^2x^3 - 3a^2x^2 + a^2x + 2abx - abdx$$

$$= \frac{1}{2}a^2x^4 – a^2x^3 + \frac{1}{2}a^2x^2 + abx^2 – abx + k.$$

(0, 0) is on the curve, $\Rightarrow$ k = 0

$\Rightarrow$ particular solution is $cy^2 + dy = \frac{1}{2}a^2x^4 – a^2x^3 + \frac{1}{2}a^2x^2 + abx^2 – abx$.

It is easy to check that (1, 0) is on the curve.

Differential Equation 3

$$(2ax – a)\ \frac{dy}{dx} = (ax^2 – ax + b)(cy + d).$$

$$\Rightarrow \int \frac{1}{cy+d}\,dy = \int \frac{ax^2 - ax + b}{2ax - a}\,dx.$$

$$\frac{1}{c}\ \ln |cy + d| = \int \frac{1}{2}x – \frac{1}{4} + \frac{b - \frac{1}{4}a}{2ax - a}\,dx.$$

$$= \frac{1}{4}x^2 – \frac{1}{4}x + ((b – \frac{1}{4}a)\frac{\ln|2ax - a|}{2a} + k.$$

(0, 0) is on the curve $\Rightarrow k = \frac{1}{c}\ln |d| – \frac{b - \frac{1}{4}a}{2a}\ \ln |–a|.$

$\Rightarrow$ particular solution is:

$$\frac{1}{c}\ \ln |cy + d| = \frac{1}{4}x^2 – \frac{1}{4}x + \frac{b - \frac{1}{4}a}{2a}\ln |2ax – a| + \frac{1}{c}\ln |d| – \frac{b - \frac{1}{4}a}{2a}\ln|–a|.$$

It is easy to check that (1, 0) is on the curve.