Risp 25: Teacher Notes

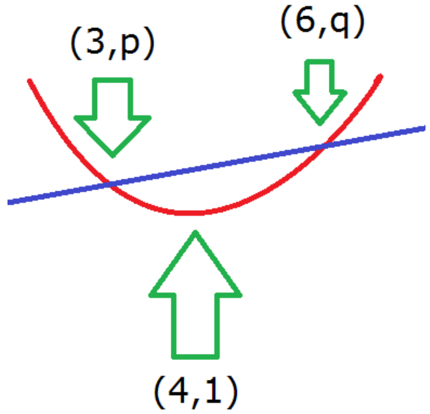
Suggested use: to consolidate/revise area between two curves,
Trapezium Rule, Volumes of Revolution, etc

It is extraordinary how much more interesting a question becomes if is opened out, thereby escaping the tyranny of the find-the-one-and-only-correct-answer approach. Compare the activity "Give the answer to 3 + 4," with "Give a question to which the answer is 7." The first is closed and fails to differentiate, whilst the second encourages a creative response, can be extended ad infinitum and can be answered at any level. It could be argued that this second question is actually a little too open: there is a delicate balance here. A totally open question can be as dull as a narrowly closed one. Perhaps saying "Give questions to which the answer is 7 that include a plus sign, a squaring and the digit 4," would be an improvement.

For this risp, students are presented each time with a picture that the topic to be studied might easily generate, but with key constants omitted. We then suppose that the answer to some natural question turns out to be really nice, say the value 1. The challenge is then to find a set of constants that will make this work.

Students may take a little time to get used to this new way of doing things. They may not initially realise that they are free to pick certain values, and that if they do this cleverly they can create a much easier question for themselves. But when they do cotton on, they will appreciate the freedom they have been given, and they will practice those key skills you wish to cultivate whilst having some wider question in view.

The area between two curves is a topic where students who know a bit about integration can be thrown straight in. They might choose the curve in the picture to be a parabola with minimum point (4,1). So its equation is $y = a(x - 4)^2 + 1$. [Students can be encouraged to use this form for the equation of the parabola before starting.]



So $p = a + 1$, $q = 4a + 1$, $m = \dfrac{q-p}{3} = a$, $c = 1 - 2a$.

So $1 = \int 36ax + 1 - 2a dx - \int 36ax^2 - 8ax + 16a + 1 dx$

Solving this is the usual way gives $a = 2/9$. Of course there are lots of other choices that will work just as well.

The Trapezium Rule question is the toughest. It is possible to find the area under the curve exactly in terms of a, b, c and d:

$$A = \int 16(ax^3 + bx^2 + cx + d)dx$$

$$= \frac{1295a}{4} + \frac{215b}{3} + \frac{35c}{2} + 5d.$$

Using the Trapezium Rule with five strips, A is approximately:

$$\frac{665a}{2} + \frac{145b}{2} + \frac{35c}{2} + 5d.$$

(Notice the terms in c and d are the same here.)

So $1 = \frac{35a}{4} + \frac{5b}{6}$, so $b = \frac{12 - 105a}{10}$, while c and d can be anything that makes the picture look right. We could choose the minimum to be at (4, 1).

$$y' = 3ax^2 + 2bx + c, \quad y'' = 6ax + 2b.$$

So $48a + 8b + c = 0$, since y' = 0 when x = 4.

In addition the gradient is increasing from 1 to 6, so y'' > 0 in that range. A sensible choice for a seems to be 0.1, which gives b = 0.15, c = -6, and d about 20.

The volumes of revolution question works out fairly neatly.

$$\int_0^1 \pi y^2 dx = \int_0^1 \pi a^2 x^{2n} dx = \frac{\pi a^2}{2n+1}$$

$$\int_0^a \pi x^2 dy = \int_0^a \pi \frac{1}{a^{2/n}} y^{2/n} dy = \frac{\pi a^n}{2+n}$$

$$\text{So } 1 = \frac{\pi a^2}{2n+1} - \frac{\pi a^n}{2+n}.$$

At this point we could choose a = 1, in which case n = 0.478..., or else maybe n = 1 (a straight line), in which case a = 1.5976... or else $n = \frac{1}{2}$ ...