Risp 22: Teacher Notes

Suggested Use: to consolidate/revise the Binomial Theorem (negative/fractional index)

While on the subject of doing and undoing, let's include this risp that once again practices a lot of useful algebra for those who persevere to the end. The 'doing' aspect is straightforward here, but the 'undoing' carries an air of mystery. Will there always be an answer? Will there ever be more than one? As a challenge, it seems a logical, neat problem that deserves an answer. Once again, the algebra treats us kindly along the way, which is always a good sign. First the doing:

$$\frac{(ax+b)^2}{c+x} = (a^2x^2 + 2abx + b^2)\left(\frac{1}{c}\right)\left(1 + \frac{x}{c}\right)^{-1}$$

$$= (a^2x^2 + 2abx + b^2)\left(\frac{1}{c}\right)\left(1 - \frac{x}{c} + \frac{x^2}{c^2} - \ldots\right)$$

$$= \frac{b^2}{c} + x\left[\frac{2ab}{c} - \frac{b^2}{c^2}\right] + x^2\left[\frac{a^2}{c} + \frac{b^2}{c^3} - \frac{2ab}{c^2}\right] + \ldots$$

This is valid if $\left|\frac{x}{c}\right| < 1$.

So the doing is not too much of a problem. Can we undo? (The algebra here is definitely not for the faint-hearted!) I am grateful to Bernard Murphy for pointing out that multiplying by (x + c) is a good idea here, giving:

$$(ax + b)^2 = pc + (p + cq)x + (q + cr)x^2\ldots$$

When using actual numbers rather than tackling the general case, this will be best. Alternatively we can use what we have done above.

If (a, b, c) is a solution, then clearly (-a, -b, c) will be a solution too. Let us not count these as different.

$$p = \frac{b^2}{c}, \text{ so } c = \frac{b^2}{p}, \text{ and } q = \frac{2ab}{c} - \frac{b^2}{c^2}$$

so substituting for c, and rearranging, we get: $a = \dfrac{b^2q + p^2}{2pb}$.

Now r = $\dfrac{a^2}{c} + \dfrac{b^2}{c^3} - \dfrac{2ab}{c^2}$,

so substituting for a and c and rearranging, we get:

$$b^4[q^2 - 4pr] + b^2[-2p^2q] + p^4 = 0.$$

This is a quadratic in b^2, and the Formula gives:

$$b^2 = \frac{p^2q \pm \sqrt{4p^5r}}{q^2 - 4pr}$$

$$\Rightarrow b^2 = \frac{p^2}{q + 2\sqrt{pr}} \text{ or } \frac{p^2}{q - 2\sqrt{pr}}.$$

So if p and r are of different sign, there will be no real solution.

But if p and r are of the same sign and $q > 2\sqrt{(pr)}$,
there will be two different solutions for b.

So if (p, q, r) = (2, 17, 8), solutions for (a, b, c) are

$$\left(\frac{13}{3}, \frac{2}{3}, \frac{2}{9}\right) \text{ and } \left(\frac{21}{5}, \frac{2}{5}, \frac{2}{25}\right).$$

This means that $\dfrac{(13x+2)^2}{9x+2}$ and $\dfrac{(21x+2)^2}{25x+2}$ both have Binomial expansions that begin $2 + 17x + 8x^2$... which is surprising, is it not?

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