Risp 20: Teacher Notes

Suggested use: to introduce/consolidate arithmetic series/sequences
sequences generally

One embarks upon a long calculation with lots of algebra. Is this worth doing? Will things just become more complicated? We are tempted to move on to something more appealing, when suddenly things begin to open up. Terms in abc cancel, it transpires that pair of terms have factors in common, and then larger factorisations become possible, at which point factors cancel, and we are left with something incomparably simpler than our starting expression. We sense that maybe here lies mathematics that is worth remembering.

As G.H. Hardy once said, "There can be no permanent place in the world for ugly mathematics." The way our algebra has simplified can be an example of such beauty, leaving us perhaps with a sense of awe. The simplification acts as a reward, reassurance that we have chosen the right path and made no mistakes. We have all had that feeling as mathematicians at some time, and it can be enormously satisfying, an 'Aha!' moment. It is an experience I am keen to pass on to my students. This risp has a couple of small 'Aha!' moments, which will hopefully give them the taste for more.

I should be careful what I claim for this risp. The solutions are obvious with the right thinking, and the way the algebra simplifies can be anticipated. It does, however, consolidate students' algebraic work within a simple and safe environment.

Students who initially know nothing about arithmetic sequences can embark straightforwardly on this activity. The problem is simple: can $S_n = u_n$? Experimentation is the best way to start, and tables can be drawn up for S_n and u_n. An Excel spreadsheet is a good idea. For the example I give, we have two possible solutions, n = 1 (which will always be a solution) and n = 13. It is possible to draw a graph of S_n against n, and another on the same axes of u_n against n. The former appears as a parabola, suggesting a quadratic formula, while the latter gives a straight line. There will clearly be at most two intersection points in this situation.

Time now for a more general treatment, if students are ready to look at arithmetic sequences more abstractly, and we can introduce standard terminology and the formula for the sum of the first n terms. This risp provides an immediate use for this. We need:

$$a + (n - 1)d = \frac{1}{2}n(2a + (n - 1)d)$$

$$\Rightarrow n^2[d] + n[2a - 3d] + [2d - 2a] = 0$$

And now our minor 'Aha!' moment: this factorises (in fact, we know it must, as n - 1 must be a factor.) So we arrive at:

$$(n - 1)(dn + (2a - 2d)) = 0 \Rightarrow n = 1$$

$$\text{or } n = \frac{2(d-a)}{d} \Rightarrow n = 2 - \frac{2a}{d}, \text{ or } d = \frac{2a}{2-n}, \text{ or } a = \frac{d(2-n)}{2}.$$

If you choose to tackle this quadratic using the formula, the simplification is even more dramatic!

Thus given a value for n and one for a, we can find a value for d so that $S_n = u_n$, and given a value for n and for d, we can find a value for a so that $S_n = u_n$.

Reflecting upon this, $S_n = u_n$ for an arithmetic series in the 'obvious' places. n = 1 will always work, and if terms cancel out exactly (as they do in the 11, 9, 7... example), then there will be a second value of n so that $S_n = u_n$.

What about geometric series? I might offer a practical example at this point, and invite experimentation: can S_n ever equal u_n here?

$$ar^{n-1} = \frac{a(r^n - 1)}{r - 1}?$$

The case r = 1 can be dealt with separately; this gives a, a, a,.... so n = 1 or a = 0.

If r is not 1, then multiplying out, we find that a = 0 or $r^{n-1} = 1$, so the only possibilities are a = 0, or n odd and r = -1. This gives the only solutions as the again 'obvious' ones, a, -a, a, -a, a,... , when $S_n = u_n$ if n is odd, and 0, 0, 0..., where r can be anything.

So far this risp has been good practice for our algebra, but we have found only ‘obvious’ examples, for our ‘obvious’ sequences, the arithmetic and the geometric ones. Can we find a sequence where $S_n = u_n$ more than twice?

We might see that $S_n = u_n$ whenever $S_{n-1} = 0$. Periodic sequences that oscillate about 0 would seem to be a good focus here.

u_n = 0, 1, 2, 1, 0, -1, -2, -1, 0... gives S_n = 0, 1, 3, 4, 4, 3, 1, 0, 0, 1... (which is periodic too), and so we have $S_1 = u_1$, $S_2 = u_2$, $S_9 = u_9$, $S_{10} = u_{10}$ ... and so on.

The Fibonacci sequence 1, 1, 2, 3, 5, 8... will clearly never have $S_n = U_n$ beyond the first term. But if we run the sequence backwards...

$$8, 5, 3, 2, 1, 1, 0, 1, -1, 2, -3, 5, -8...$$

and start our sequence at 0 (giving 0, 1, -1, 2, -3, 5, -8...) then $S_n = u_n$ for n = 1, 2, 4.

What happens if we say $u_n = a_m n^m + a_{m-1} n^{m-1} + ... a_0$? Let’s say:

$$u_n = an^3 + bn^2 + cn - 1.$$

Putting $S_n = 0$ for n = 2 to 4 gives a = $\frac{1}{15}$ b = $-\frac{11}{20}$, c = $\frac{83}{60}$, so $S_n = u_n$ for n = 1, 3, 4 and 5.

We can clearly extend this for polynomials of higher degree.

And finally; we could write down the digits of π, making them alternately positive and negative. The random nature of π’s digits suggest that we will arrive at $S_{n-1} = 0$ infinitely often.

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