Risp 18: Teacher Notes

Suggested Use: To consolidate/revise the composition of functions

Like many, maybe most, of these risps, this one came about mid-lesson, and was suggested by a student comment. Asking when fg = gf is the most natural of questions, but in many years of teaching composition of functions, I had never explored it. One of the marvellous things about teaching mathematics is that whatever the level you teach at, there are always new ways to look at old ideas, and you can always trust your students to find these new ways.

Suppose we choose:

$$f(x) = x^2 + 1, \text{ and } g(x) = 2x + 4$$
$$\Rightarrow fg(x) = (2x + 4)^2 + 1 = 4x^2 + 16x + 17$$
$$\Rightarrow gf(x) = 2(x^2 + 1) + 4 = 2x^2 + 6.$$

So fg(x) does not equal gf(x) for all values of x.

Can we find any x so that $2x^2 + 6 = 4x^2 + 16x + 17$?

This gives $2x^2 + 16x + 11 = 0$, which gives x = -0.760 or -7.24. So fg(x) = gf(x) for two values of x.

If f = g, then fg = gf, whatever f and g are.

If f(x) = x, then fg = gf whatever g is.

If $f(x) = x^n$ and $g(x) = x^m$, then fg always equals gf.

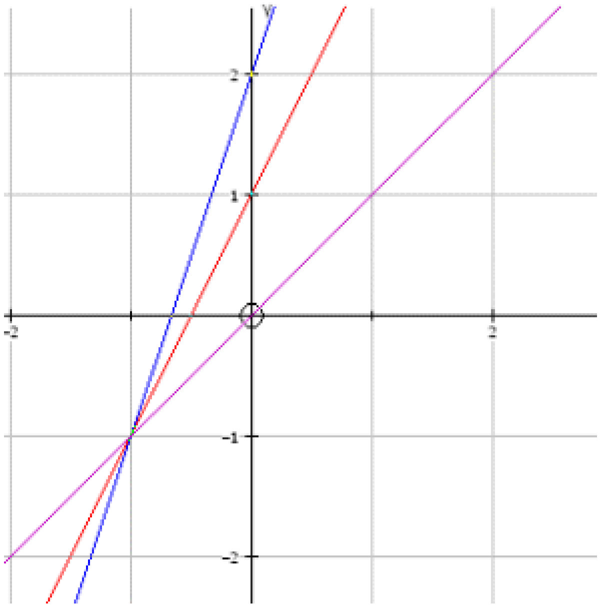
Asking whether fg = gf when both f and g are linear seems like the kind of question that will lead to a trivial solution. But no:

$$f(x) = ax + b,\ g(x) = cx + d$$

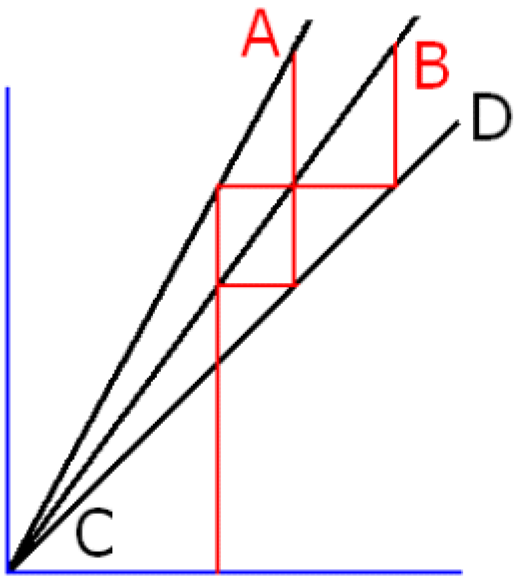
$$\Rightarrow fg(x) = a(cx + d) + b = acx + ad + b,$$
$$\text{and } gf(x) = c(ax + b) + d = acx + bc + d$$

$$\text{So } fg = gf \Rightarrow ad + b = bc + d \Rightarrow d = \frac{b(c-1)}{a-1}.$$

So if we choose a, b and c to be anything (excepting a = 1), we can find a value for d that will mean fg = gf. Where do y = f(x) and y = g(x) intersect in this case?
ax + b = cx + b(c - 1)/(a - 1) gives x = b/(1 - a), y = b/(1 - a); in other words, the two lines intersect on the line y = x.



This gives us the following 'theorem':



If the line CD is of gradient 1, then the line AB will be horizontal.

Now comes the chance to get the stronger students to use their imagination to find other functions that commute.

If f = $\dfrac{ax+1}{x+b}$ and g(x) = $\dfrac{cx+1}{x+d}$, then

fg(x) = $\dfrac{(ac+1)x+a+d}{(b+c)x+bd+1}$ and gf(x) = $\dfrac{(ac+1)x+b+c}{(a+d)x+bd+1}$.

So fg = gf $\Leftrightarrow$ a + d = b + c, or a - b = c - d.

This can lead on to some very deep questions. Given two Mobius transformations f(z) and g(z), when does f(g(z)) = g(f(z))? Advanced stuff indeed. The discussion of when two functions commute is meat and drink to group theorists. I am grateful to Ian Short for supplying the insights below:

Let us work within a group G. Given f in G, the set of all elements g in G that satisfy fg = gf together form a subgroup of G, called the centraliser of f. In other terms, $g^{-1}fg = f$, so the elements g are the elements that conjugate f to itself.

A concrete example: consider the symmetries of a regular n-gon. These consist of rotations and reflections. Let r be a rotation that is not the identity map. This commutes with all other rotations (i.e. fr = rf for every other rotation f). On the other hand, for any reflection f, we have that frf = r^{-1}, so f and r only commute if r = r^{-1} (that is, r is a rotation by 180 degrees).

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