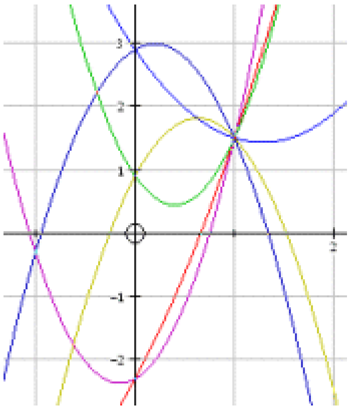
Risp 17: Teacher Notes

Suggested use: to consolidate/revise quadratic curves

This is a risp that relies on technology, more specifically on the Constant Controller facility in your graphing package. In a sense we create a little 'microworld' with these six parabolas, and their three parameters a, b and c. Prior knowledge about quadratic curves is extremely helpful in tackling the questions posed here successfully, but for those whose theory is skimpy, there is the chance to simply play. Perhaps this is the nearest thing to a computer game that this collection of risps has produced so far.

What do the parabolas have in common? Well, a point for starters. (1, k) for some value k lies on all the curves, and one would hope most students can swiftly twig that k is a + b + c.



Managing a, b and c so that all six lines of symmetry are to the left of the y-axis is easy. (In fact, the starting position gives a trivial case of this.) Getting all the lines of symmetry to lie to the right of the y-axis proves infuriatingly difficult, and a little thought combined with some theory reveals this to be impossible. We need:

$$\frac{-a}{2b},\ \frac{-b}{2c} \text{ and } \frac{-c}{2a} \text{ to all be positive.}$$

But if a and b are of different sign, and b and c are of different sign, then c and a must be of the same sign, so $\frac{-c}{2a}$ must be negative.

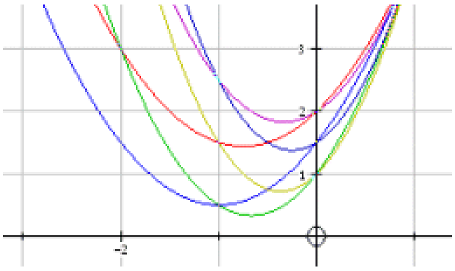
Can we arrange things so that all six parabolas have two solutions for y = 0? We can. We need:

$$a^2 > 4bc,\ b^2 > 4ac,\ c^2 > 4ab,$$

and choosing a and b to be positive and c to be large and negative means this is certainly possible.

This gives twelve crossings of the x-axis.

No solutions? Now we need $a^2 < 4bc$, $b^2 < 4ac$, $c^2 < 4ab$. Clearly a, b and c need to be all positive or all negative. (1, 2, 3) fails, but (1, 1.5, 2) succeeds. (-1, -1.5, -2) will do just as well. (In fact, the original starting position will do, but let us hope our students will have forgotten about that by now!)



For a parabola to touch the x-axis, we need, say, $b^2 = 4ac$. But this gives two parabolas that touch the x-axis, $y = ax^2 + bx + c$, and $y = cx^2 + bx + a$. The closest we can get to a single parabola touching the x-axis is if a = c. Should two parabolas on top of each other count as a single parabola? An interesting discussion point.

The lines x = 0 and x = -1 have the property that three pairs of parabolas always seem to intersect upon them. (See Fig. 3) The algebra behind this is pleasing. Suppose we take $y = ax^2 + bx + c$. Where does this intersect with:

i) $y = ax^2 + cx + b$? At (1, a + b + c).
ii) $y = bx^2 + ax + c$? At (0, c) and (1, a + b + c).
iii) $y = bx^2 + cx + a$? At $\left(\frac{a-c}{a-b}, \ldots\right)$ and (1, a + b + c).
iv) $y = cx^2 + ax + b$? At $\left(\frac{b-c}{a-c}, \ldots\right)$ and (1, a + b + c).
v) $y = cx^2 + bx + a$? At (-1, a - b + c) and (1, a + b + c).

It is clear from this that if a, b and c are rational, then the coordinates of crossing points will be rational too.

Think of a property of quadratic curves that you would like to test or explore, then think of an aspect of this risp that could generate, examine, require this property in some way. My feeling is that this is a genuinely rich situation, one where many more interesting questions could arise quite naturally.

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