Risp 15: Teacher Notes

Suggested use: to consolidate/revise circles in coordinate geometry

I have earned a crust in an A Level maths classroom for more years than I dare to remember now, but there are still moments where I am caught out. I was going through a circles-in-coordinate-geometry problem with the whole class the other day, and I found myself saying something I immediately doubted (this can always be taken as a Risp Alert!) What to do? Explore the glitch in my thinking there and then, or keep the thread of the lesson going? On this occasion I did the latter, but I did return to the thought that had discomfited me afterwards. And sure enough, there was a risp to be found.

A few lessons later, I tried the resultant activity with the same group. Prior knowledge? They had met the equation of a circle in the form $(x - a)^2 + (y - b)^2 = r^2$. They'd done a little work expanding this and had examined the result, going on to classify various second-order curves into circles and non-circles. How confident and flexible was their understanding? This risp would be a good test.

Suppose we start by picking the numbers 3, 2 and 1. Then we create the following six equations:

1. $x^2 + y^2 + 3x + 2y + 1 = 0$
2. $x^2 + y^2 + 3x + y + 2 = 0$
3. $x^2 + y^2 + 2x + 3y + 1 = 0$
4. $x^2 + y^2 + 2x + y + 3 = 0$
5. $x^2 + y^2 + x + 2y + 3 = 0$
6. $x^2 + y^2 + x + 3y + 2 = 0$

Your graphing package will give circles for numbers 1, 2, 3 and 6, while Numbers 4 and 5 produce nothing at all. I allowed the group half an hour trying out various starting triplets, including a triplet of negatives, and a triplet of numbers between 0 and 1. The exploration over for the moment, we left the computers and returned to our classroom, ready to tackle some theory. I drew up two columns, labelled 'Facts' and 'Conjectures', and asked for contributions. I requested things that people felt sure about as 'Facts', and possible truths where there remained some doubt for 'Conjectures' – in reality, given that we had not tried to prove anything yet, the distinction between 'Facts' and 'Conjectures' was a little blurred. The 'Facts' I received were these:

1. You get a circle most of the time.
2. Otherwise you get nothing. If this happens, the largest of your three numbers is last.
3. Choosing three negative numbers always gives a full set of six circles.
4. Choosing three numbers between 0 and 1 can give six circles, and can give no circles at all.

'Conjectures' were harder to come by. After a lot of thought, one group in effect offered:

"You get a full set of six circles when $0 < a < b < c \Leftrightarrow c < a + b$."

Then a second group said:

"If (a, b, c) is a Pythagorean Triple, then you get a full set of six circles."

Where had these come from? My students had not tried to write much down in the computer room. These hypotheses were both based almost purely on their raw mathematical intuition. So were their 'guesses' inspired hits or wild misses? The time had come to introduce the theory as I saw it.

Given $x^2 + y^2 + ax + by + c = 0$, we can rewrite this as $x^2 + ax + y^2 + by + c = 0$.

Completing the square: $(x + \frac{a}{2})^2 - \frac{a^2}{4} + (y + \frac{b}{2})^2 - \frac{b^2}{4} + c = 0$.

So $(x + \frac{a}{2})^2 + (y + \frac{b}{2})^2 = \frac{a^2 + b^2 - 4c}{4}$.

So this is (maybe) a circle with centre $(-\frac{a}{2}, -\frac{b}{2})$

and radius $r = \frac{1}{2}\sqrt{a^2 + b^2 - 4c}$.

So the circle exists $\Leftrightarrow a^2 + b^2 > 4c$.

We were able now to examine the truth or falsity of our 'Facts.' If c is negative, the circle always exists, so if a, b, and c are all negative, we get a full set of six circles.

Choosing 3, 2, and 1 gives us only four out of six. But choosing 10, 11, and 12 gives a full set of six circles. If a, b and c are all between 0 and 1, we may get no circles at all. But choosing 1/2, 1/3, and 1/32 gives two circles. Three numbers between 1/2 and 1 always give no circles.

How do our two conjectures look in the light of this? The first, that $c < a + b$ is the condition for six circles, should read $c < \frac{a^2 + b^2}{4}$, a good attempt in my book.

And the second? If we have a Pythagorean Triple, then $a^2 + b^2 = c^2 > 4c$, which is true, since the smallest c can be is 5. So the second conjecture is true! Which left me a little amazed. The class were impressed by both conjectures, one along the right lines, the other a theorem, resulting in spontaneous applause for their authors.

www.risps.co.uk