Risp 12: Two Repeats

Teacher notes

Suggested Use: to consolidate/revise algebra

Syllabus areas covered:
Changing the subject of a formula
Graphical solution of equations
Simultaneous equations
Solving a quadratic
Manipulating surds

Sometimes there is a lesson that is a bit different for some reason. Maybe half the class have been entered for an exam, or the snow has drifted that day. I find it is always a good idea to have activities up your sleeve to cover for this. Activities like this risp, a piece of mathematical circuit training that consolidates useful skills without requiring any extra theory that your absentees will be missing. By the end of this task, your students will know they have been given a thorough algebraic workout.

So how might we start here? Students sometimes lose confidence in the face of a novel question such as this. How many possible pairs are there? It looks as if there might be $^6C_2$ = 15 possible pairs, leading on to ($^6C_2$ x $^4C_2$)/2 = 45 pairs of pairs. The idea of checking out 45 possibilites makes the keenest mathematician's heart sink. But of course, we can start to rule out lots of these quite quickly.

Immediately we can see that x - y is the only one of the six numbers

that is negative, so it cannot be a repeat.

Can x + y = y - x? This gives x = 0, which is not allowed.

Can xy = x/y? This implies y = ±1, which again is a contradiction.
Similarly for xy = y/x.
Can y/x = x/y? Again, the contradiction is clear.

So the five remaining numbers fall into two groups,
{x + y, y - x} and {xy, x/y, y/x}.

The first repeat must be x + y = one of the second set,
and the second repeat must be y - x = another one of the second set.

What if x + y = xy? x + y is bigger than x, while xy is smaller than x.

Yet another contradiction.

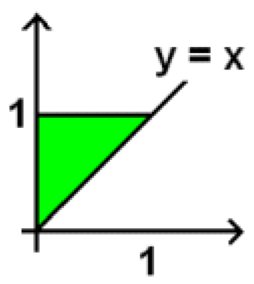
And if y - x = y/x, we have the problem that y - x is less than 1 while y/x > 1.

So we end up with just four pairs that might be true:

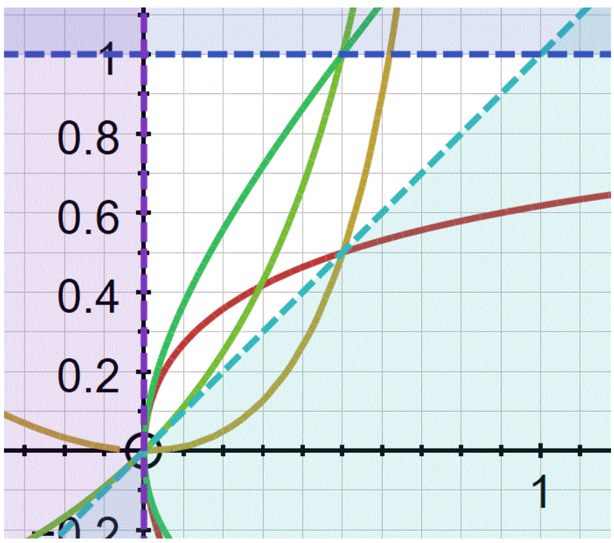
1. x + y = x/y, 2. x + y = y/x, 3. y - x = xy, and 4. y - x = x/y.

A much less daunting number than the 15 we started out with.

Where to now? Why not bring in our graphing package to draw these four curves? (Asking students to first change the subject to either x or y in each case provides a chance for some more excellent practice!) What kind of crossing points are we looking for? We must have two of the curves intersecting in the following region (boundary not included):



This is the picture your package will draw:



It is easy to read off the only two curves that meet within the region we are looking for (meeting on the boundary is no good):

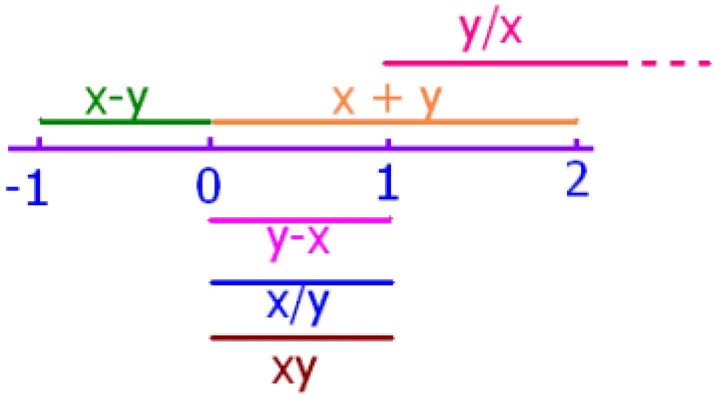
x + y = x/y and y - x = xy.

Solving these together as simultaneous equations eventually gives

$$x = 1 - \frac{1}{\sqrt{2}}, \text{ and } y = \sqrt{2} - 1.$$

If your students are not completely algebra-ed out by this stage, they can put these answers back into x + y, xy, x/y, y/x, x - y and y - x, and check that they do indeed give two repeats. This is excellent surd practice.

Another approach is to map out on a single diagram what ranges the six initial numbers might fall into.



We end up with four possible combinations to explore:

1. x + y = y/x, y - x = xy
2. x + y = y/x, y - x = x/y
3. x + y = x/y, y - x = y/x
4. x + y = x/y, y - x = xy

Multiplying the two equations in 1 gives $y^2-x^2 = y^2$ (contradiction).
Doing the same for 2 or 3 gives $y^2-x^2 = 1$, or $y^2>1$ (contradiction).
Which leaves 4 as the only viable option, yielding our unique numbers.

Overall, a problem with no excessively demanding theory that covers a remarkably wide range of algebraic techniques. It can be seen as the little brother of some of the advanced and nasty problems that go

Person A says, "I do not know the numbers are,"

to which B replies, "Now I know what the numbers are!"

Your absentees will hopefully arrive back to find their colleagues more confident and skilled in the basics.