Risp 11: Teacher Notes

Suggested use: To consolidate the Remainder/Factor Theorems

Sometimes less is more with risps. The activity above is really simple, yet it proved happily effective with my AS students. This starter once again provides good practice, here in the use of the Remainder Theorem, with proving a conjecture as an end point in view to provide a motivation. The final result when viewed as a theorem might be scorned as ordinary, yet the proof of the Remainder Theorem itself can look fairly trivial at times, while it remains surprisingly useful.

Suppose a student chooses 31 and 19 as their integers. (The exercise will work with numbers that are close together, but if they are too close it is harder to work out what is happening.) Let's say they then pick $x^3 + x - 1$ as their polynomial.

Dividing by $(x - 31)$ gives a remainder of $31^3 + 31 - 1$, or 29821.

Students can try working with some pretty big numbers here, which increases the drama.

Dividing by $(x - 19)$ gives a remainder of $19^3 + 19 - 1$ or 6877.

So $R_a - R_b = 22944$, $a - b = 12$, and $\frac{22944}{12} = 1912$ exactly.

So do we always get the remainder zero? Students soon compare notes on this, and wonder how to show this natural conjecture is true. It is easiest here to start with the general quadratic as the chosen polynomial.

Starting with $px^2 + qx + r = f(x)$, the remainder on dividing $f(x)$ by $(x - a)$ is $pa^2 + qa + r$, and on dividing $f(x)$ by $(x - b)$ is $pb^2 + qb + r$.

$$\text{So } R_a - R_b = p(a^2 - b^2) + q(a - b) = (a - b)(pa + pb + q).$$

So $(a - b)$ is a factor of $R_a - R_b$.

Is $(a - b)$ a factor of $a^3 - b^3$? Of $a^4 - b^4$? Students can be encouraged to pattern-spot here. To generalise to all polynomials, we need to show that $(a - b)$ is a factor of $a^n - b^n$. One of the pleasing things about this risp is that this is simply a consequence of the Factor Theorem; $a - b$ is zero when $a = b$, and putting $a = b$ into $a^n - b^n$ gives zero, so $a - b$ must be a factor of $a^n - b^n$. There is a sense of having come full circle.

While on the topic of factorising $a^n - b^n$; will the various terms $a^k b^j$ in any factors always have coefficients of 1 or -1? Try putting $a^{105} - b^{105}$ into a computer algebra package...