Risp 1: Teacher Notes

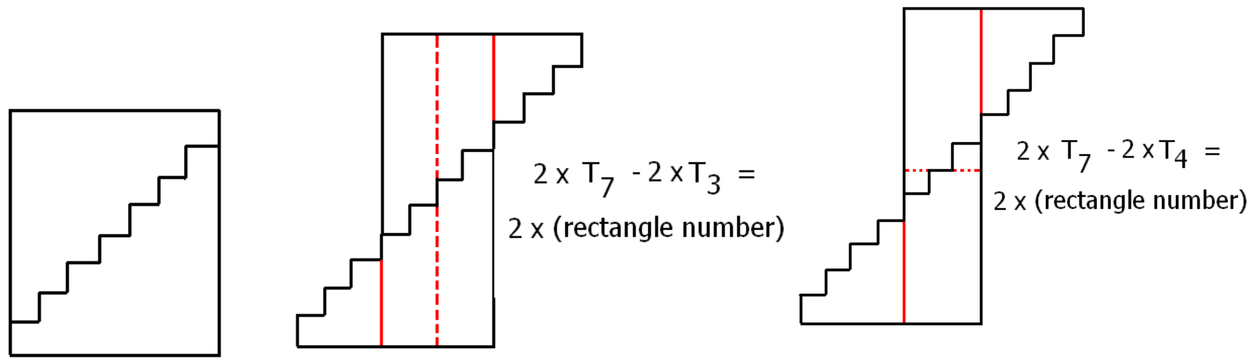
Suggested use: to introduce/consolidate/revise ideas of proof

Skills included:
to be able to construct and present a mathematical argument
appropriate use of logical deduction

A mathematician can't help but see things differently. Reflect on the following photo that I took whilst enjoying an all-to-brief visit to Barcelona a few years ago:

Monument to Francesc Macia, Place Catalunya, by Josep Maria Subirachs

Do you sense a piece of mathematics trying to emerge from this statue? I got back to the hotel and started to sketch a few staircases.



In other words, $T_{n+m} - T_n$ is going to be a rectangle number (a composite number) - EXCEPT? When might this fail?

After a few hours, I'd managed to hone the problem into a form that I felt I could present to a class. After running it with students a few times, I ended up with this risp.

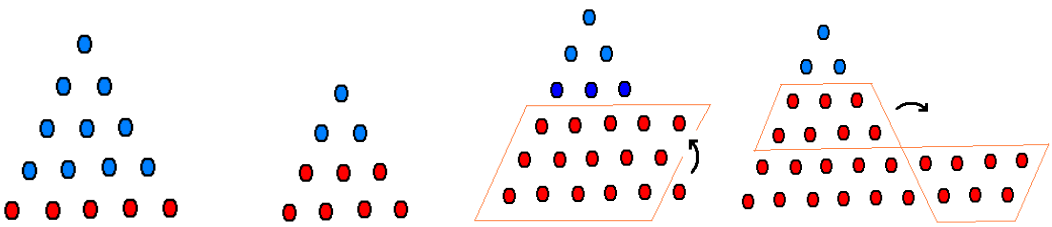
All that is required in terms of prior knowledge here is an understanding of 'prime number' and 'triangle number'. As students begin work on this problem, it is sensible to suggest they draw up a table to get themselves started:

	1	3	6	10	15	21	28	36	45	55
1	0	2	5	9	14	20	27	35	44	54
3		0	3	7	12	18	25	33	42	52
6			0	4	9	15	22	30	39	49
10				0	5	11	18	26	35	45
15					0	6	13	21	30	40
21						0	7	15	24	34
28							0	8	17	27
36								0	9	19
45									0	10
55										0

Some students might note that the primes fall close to the main diagonal.

"So the only chance for $T_{n+m} - T_n$ to be prime is if m is 1 or 2."

Can we find an argument to back this up? This proof-without-words works well:



So the bottom row can be prime, the bottom two rows can be prime, but try to take a larger number of rows, and you will always get a rectangle number.

If your students are up for a little algebra, then they might emerge with this:

$$T_{n+m} - T_n = \frac{(n+m)(n+m+1)}{2} - \frac{n(n+1)}{2} = \frac{m(2n+m+1)}{2}$$

which can only be prime if m = 1 or 2.

The extension material is straightforward from here.

$(n + m)^2 - n^2 = (2n + m)m$, which can only possibly be prime if m = 1.

$(n + m)^3 - n^3 = (n^2 + 3mn + 3m^2)m$, which can only possibly be prime if m = 1.

My thanks to Bernard Murphy for pointing out that the difference of two fourth powers factorises neatly to show that it's never prime.